B.Sc Part II (Hons) Paper III

FACTOR GROUP (QUOTIENT GROUP).

Definition: — Let G be a group and let H be a normal subgroup of G. Then the group of cosets of H in G w.r.t multiplication

$$xH \cdot yH = xyH : x, y \in G$$

is called the factor group (or quotient group) of G with respect to H and it is denoted by G/H. Remember that G/H is the collection of all the left (right) cosets of H in G.

Theorem: — Let G be a group and let H be a normal subgroup of G. Then the set of cosets of H in G is a group w.r.t the multiplication

$$xH \cdot yH = xyH. \ x, y \in G.$$

Proof: — Here the elements of the set say S, are

$eH, xH, yH, zH,$ — i.e. $S = \{aH : a \in G\}$.

We verify the group axioms one by one.

(i) closure property: If $x, y \in G$ then $xH \cdot yH = xyH$ and therefore $\in S$

since if $x, y \in G$, then $xy \in G$

(ii) Associativity: If $x, y, z \in G$, then we write have

$(xH \cdot yH) \cdot zH = (xyH) zH = (xy) zH$

$xH \cdot (yH \cdot zH) \ xH \cdot (yzH) = x(yz)H$

Since $(xy)z = x(yz)$

Therefore $(xH \cdot yH) zH = xH \cdot (yH \cdot zH)$

(iii) Existence of identity: The coset $eH = H$ where e is the identity in G, is the identity S, since

$eH \cdot xH = exH = xH.$

Existence of inverse: The inverse of the coset xH is $x^{-1}H$ since $xH \cdot x^{-1}H = (xx^{-1})H = eH$

Thus the set S is a group.

Theorem: Every factor group of an Abelian group is Abelian

Proof – Let G be an Abelian group and N be a subgroup of G

Then N is a normal subgroup of G. We are required to prove that G/N is Abelian. If $x, y \in G$, then xN, yN are any two elements of G/N. We have

$(xN)(yN) = xyN = yxN$

$\because G$ is Abelian $\therefore xy = yx$

$= (yN)(xN)$

$\therefore G/N$ is Abelian.

But the converse is not true. This means that even if G is not Abelian, G/N can be Abelian. For example, if P_3 be the symmetric group of degree 3 and A_3 be the alternating group of order 3, then P_3/A_3 is an Abelian group while P_3 is not an Abelian group. The group P_3/A_3 is of order 2 and every group of order 2 is Abelian.

Theorem: Show that every factor group (quotient group) of a cyclic group is cyclic

Proof – Let G be a cyclic group and a be a generator of G. Let H be a subgroup of G. Since every cyclic group is Abelian, therefore H is a normal subgroup of G. Let a^k be any element of G where k is some integer. Then Ha^k is any element of G/H. We are going to show that $Ha^k = (Ha)^k$ for every integer k. We have $Ha^k = H(aaa \text{---} \text{to } k \text{ times})$

$= (Ha)(Ha)(Ha) \text{———} k \text{ times} = (Ha)^k$

Therefore G/H is a cyclic group and Ha is [illegible]

[illegible]